Inequalities: Exercises 1A.

1. If $a_1, a_2, \ldots, a_n$ be n positive real numbers in ascending order of magnitude, prove that

$$a_1 < \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1 + a_2 + \cdots + a_n} < a_n$$

Solution: Given that $a_1 < a_2 < \cdots < a_n$.

Now, for $i = 2, 3, \ldots, n$,

$$\begin{aligned} a_1 < a_i &\implies a_1 a_i < a_i^2 \; [\because \text{ each } a_i > 0] \\ &\implies a_1(a_1 + a_2 + \cdots + a_n) < a_1^2 + a_2^2 + \cdots + a_n^2 \\ &\implies a_1 < \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1 + a_2 + \cdots + a_n} \qquad (1) \end{aligned}$$

Again, for $i = 1, 2, 3, \ldots, n-1$,

$$\begin{aligned} a_i < a_n &\implies a_i^2 < a_n a_i \; [\because \text{ each } a_i > 0] \\ &\implies a_1^2 + a_2^2 + \cdots + a_n^2 < a_n(a_1 + a_2 + \cdots + a_n) \\ &\implies \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1 + a_2 + \cdots + a_n} < a_n \qquad (2) \end{aligned}$$

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From (1) and (2), we get

$$a_1 < \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1 + a_2 + \cdots + a_n} < a_n$$

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2. If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, and $p_1, p_2, \ldots, p_n$ be positive real numbers, prov that

$$\min(a) < \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{(p_1 + p_2 + \cdots + p_n)} < \max(a),$$

where $\min(a)$ and $\max(a)$ respectively denote the least and greatest of a_i for $i = 1, 2, \ldots, n$

Solution: Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, and $p_1, p_2, \ldots, p_n$ be positive real numbers. Also let $\min(a)$ and $\max(a)$ respectively denote the least and greatest of a_i for $i = 1, 2, \ldots, n$.

Then $\min(a) \leq a_i$ and $a_i \leq \max(a)$ for $i = 1, 2, \ldots, n$.

Since $a_1, a_2, \ldots, a_n$ are not all equal, $\exists$ integers $j, k (1 \leq j, k \leq n)$ such that $\min(a) < a_j$ and $a_k < \max(a)$.

Now,

$$\begin{aligned} \min(a) \leq a_i &\implies p_i \min(a) \leq p_i a_i \; [\because p_i > 0] \\ &\implies (p_1 + p_2 + \cdots + p_n) \min(a) < p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \; [\because \min(a) < a_i] \\ &\implies \min(a) < \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{(p_1 + p_2 + \cdots + p_n)} \; [\because \text{each } p_i > 0] \end{aligned} \tag{3}$$

Again,

$$\begin{aligned} a_i \leq \max(a) &\implies p_i a_i \leq \max(a) p_i \; [\because p_i > 0] \\ &\implies p_1 a_1 + p_2 a_2 + \cdots + p_n a_n < \max(a)(p_1 + p_2 + \cdots + p_n) \; (\because a_k < \max(a)) \\ &\implies \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{(p_1 + p_2 + \cdots + p_n)} < \max(a) \; [\because \text{each } p_i > 0] \end{aligned} \tag{4}$$

Combining (3) and (4), we get

$$\min(a) < \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{(p_1 + p_2 + \cdots + p_n)} < \max(a)$$

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3. If $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all real numbers and $b_i > 0$ for $i = 1, 2, \ldots, n$, prove that

$$m \leq \frac{a_1 + a_2 + \cdots + a_n}{(b_1 + b_2 + \cdots + b_n)} \leq M,$$

where $m = \min\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\}, M = \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\}$.

Solution:

$$\begin{aligned} m = \min\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\} &\implies m \leq \frac{a_i}{b_i}, \forall i = 1, 2, \ldots, n \\ &\implies m b_i \leq a_i, \forall i = 1, 2, \ldots, n \\ &\implies m(b_1 + b_2 + \cdots + b_n) \leq a_1 + a_2 + \cdots + a_n \\ &\implies m \leq \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \; [\because \text{each } b_i > 0] \end{aligned} \tag{5}$$

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Again,

$$M = \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\} \implies \frac{a_i}{b_i} \leq M, \forall i = 1, 2, \dots, n$$
$$\implies b_i \leq M a_i, \forall i = 1, 2, \dots, n$$
$$\implies b_1 + b_2 + \cdots + b_n \leq M(a_1 + a_2 + \cdots + a_n)$$
$$\implies \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq M \; [\because \text{each } b_i > 0] \tag{6}$$

Combining (5) and (6), we get

$$m < \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq M$$

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4. If $x_1, x_2, \dots, x_n$ be n real numbers satisfying $0 < x_1 < x_2 < \cdots < x_n < \frac{\pi}{2}$, prove that

$$\tan x_1 < \frac{\sin x_1 + \sin x_2 + \cdots + \sin x_n}{\cos x_1 + \cos x_1 + \cdots + \cos x_n} < \tan x_n.$$

Solution:

$$0 < x_1 < x_2 < \cdots < x_n < \frac{\pi}{2}$$
$$\implies 0 < \sin x_1 < \sin x_2 < \cdots < \sin x_n < 1 \; [\because \text{`}\sin\text{' function is increasing on } (0, \pi/2)]$$
$$\implies \sin x_1 < \sin x_i, \text{for all } i = 2, 3, \dots, n \text{ and } \sin x_i < \sin x_n, \text{for all } i = 1, 2, \dots, n-1$$
$$\implies n \sin x_1 < \sin x_1 + \sin x_2 + \cdots + \sin x_n \;\&\; \sin x_+ \sin x_2 + \cdots + \sin x_n < n \sin x_n$$
$$\implies n \sin x_1 < \sin x_1 + \sin x_2 + \cdots + \sin x_n < n \sin x_n \tag{7}$$

Again,

$$0 < x_1 < x_2 < \cdots < x_n < \frac{\pi}{2}$$
$$\implies 1 > \cos x_1 > \cos x_2 > \cdots < \cos x_n > 0 \; [\because \text{`}\cos\text{' function is decreasing on } (0, \pi/2)]$$
$$\implies \cos x_1 > \cos x_i, \text{for all } i = 2, 3, \dots, n \text{ and } \cos x_i > \cos x_n, \text{for all } i = 1, 2, \dots, n-1$$
$$\implies n \cos x_1 > \cos x_1 + \cos x_2 + \cdots + \cos x_n \;\&\; \cos x_1 + \cos x_2 + \cdots + \cos x_n > n \cos x_n$$
$$\implies \frac{1}{n \cos x_1} < \frac{1}{\cos x_1 + \cos x_2 + \cdots + \cos x_n} \;\&\; \frac{1}{\cos x_1 + \cos x_2 + \cdots + \cos x_n} < \frac{1}{n \cos x_n}$$
$$\implies \frac{1}{n \cos x_1} < \frac{1}{\cos x_1 + \cos x_2 + \cdots + \cos x_n} < \frac{1}{n \cos x_n} \tag{8}$$

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From (7) and (8)

$$\tan x_1 < \frac{\sin x_1 + \sin x_2 + \cdots + \sin x_n}{\cos x_1 + \cos x_2 + \cdots + \cos x_n} < \tan x_n$$

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5. If a, b, c be real numbers, prove that

$$(a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2 \geq ab + bc + ca.$$

Solution:

$$(a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2 = 3(a^2+b^2+c^2) - 2(ab+bc+ca) \tag{9}$$

$(a^2+b^2) \geq 2ab, (b^2+c^2) \geq 2bc, (c^2+a^2) \geq 2ca$ equality occurs when $a = b = c$
Adding these, we get $(a^2+b^2+c^2) \geq (ab+bc+ca)$. Using this result in (9), we have

$$\begin{aligned}(a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2 &\geq 3(ab+bc+ca) - 2(ab+bc+ca) \\ &= ab+bc+ca\end{aligned}$$

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6. If a, b, c be positive real numbers such that the sum of any two is greater than the third, prove that

   (a) $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) \geq 0$ for all real p, q, r;

   (b) $a^2yz + b^2zx + c^2xy \leq 0$ for all real x, y, z such that $x + y + z = 0$.

Solution:

(a) **Case I:** Let $p = q = r$.
Then $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) = 0$.

Case II: Let $p = q \neq r$.
Then $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) = 0 + 0 + c^2(p-r)^2 > 0$.

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Case III: Let $p \neq q \neq r$. Without loss of generality, let $p > q > r$.

Then $q - p < 0$ & $q - r > 0 \implies (q-p)(q-r) < 0$. Given $b < c + a$.

From given condition,

$$\begin{aligned} b < c + a &\implies b^2 < (c+a)^2 \\ &\implies b^2(q-p)(q-r) > (c+a)^2(q-p)(q-r)\,[\because (q-p)(q-r) < 0] \end{aligned}$$

Now,

$$\begin{aligned} & a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) \\ & > a^2(p-q)(p-r) + (c+a)^2(q-p)(q-r) + c^2(r-p)(r-q) \\ & > a^2(p-q)(p-r) + 2ca(q-p)(q-r) + c^2(r-p)(r-q)\,[\because (c+a)^2 > 2ca] \\ & > a^2(p-q)(p-q) + 2ca(q-p)(q-r) + c^2(r-p)(r-q)\,[\because p-r > p-q] \\ & > a^2(p-q)^2 - 2ca(p-q)(q-r) + c^2(p-r)(q-r) \\ & > a^2(p-q)^2 - 2ca(p-q)(q-r) + c^2(q-r)^2\,[\because p-r > q-r] \\ & = (a(p-q) - c(q-r))^2 \\ & \geq 0 \end{aligned}$$

Alternatively, Case I: Let $p = q = r$.

Then $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) = 0$.

Case II: Let $p = q \neq r$.

Then $a^2(p-q)(p-r) + b^2(q-p)(q-r) + c^2(r-p)(r-q) = 0 + 0 + c^2(p-r)^2 > 0$.

Case III: Let $p \neq q \neq r$. Without loss of generality, let $p > q > r$.

Then $p - q = x > 0, q - r = y > 0$ and $r - p = z < 0$. Also $x + y + z = 0$.

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\begin{aligned}
\text{L.H.S.} &= a^2(p-q)(p-r)+b^2(q-p)(q-r)+c^2(r-p)(r-q)\\
&= a^2x(-z)+b^2(-x)y+c^2z(-y)\\
&= -(a^2xz+b^2xy+c^2yz)\\
&= -[a^2x(-x-y)+b^2xy+c^2y(-x-y)]\\
&= a^2x^2+a^2xy-b^2xy+c^2xy+c^2y^2\\
&= a^2x^2+c^2y^2+xy(c^2+a^2-b^2)\\
&= (ax-cy)^2+2acxy+xy(c^2+a^2-b^2)\\
&= (ax-cy)^2+(2ac+c^2+a^2-b^2)\\
&= (ax-cy)^2+[(c+a)^2-b^2]\\
&= (ax-cy)^2+[(c+a+b)(c+a-b)]\\
&> 0\,[\because (c+a-b)>0].
\end{aligned}
$$

(b) Given that $x+y+z=0$, we have two of them to be of different sign. Without loss of generality, let us assume x and y have different sign, i.e., $xy \le 0$.

We have $z = (x+y)$ and hence

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\begin{aligned}
a^2yz+b^2zx+c^2xy &= c^2xy-(x+y)(a^2y+b^2x)\\
&= xy(c^2-a^2-b^2)-a^2x^2-b^2y^2\\
&= xy(c^2-a^2-b^2+2ab)-(ax+by)^2\\
&= xy(c^2-(a-b)^2)-(ax+by)^2\\
&= xy(c+a-b)(c-a+b)-(ax+by)^2\\
&\le 0\,[\because xy<0, xy(c+a-b)(c-a+b)<0]
\end{aligned}
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7. If a, b, c are positive real numbers, each less than 1, prove that

$$8(abcd+1) > (a+1)(b+1)(c+1)(d+1).$$

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Solution: Since, $a < 1, b < 1, c < 1, d < 1 \implies 1-a > 0, 1-b > 0, 1-c > 0, 1-d > 0$. Now,

$$\begin{aligned}(1-a)(1-b) > 0 &\implies 1-(a+b)+ab > 0\\ &\implies 1+ab > a+b\\ &\implies 1+ab+1+ab > a+b+1+ab\\ &\implies 2(1+ab) > (1+a)(1+b) \qquad (10)\end{aligned}$$

Similarly, we get

$$2(1+cd) > (1+c)(1+d) \qquad (11)$$

From (10) & (11), we get

$$4(1+ab)(1+cd) > (1+a)(1+b)(1+c)(1+d) \qquad (12)$$

Again, $a < 1, b < 1, c < 1, d < 1 \implies ab < 1, cd < 1$. By previous result we can say

$$2(1+abcd) > (1+ab)(1+cd) \qquad (13)$$

Hence, $8(1+abcd) > (1+a)(1+b)(1+c)(1+d)$.

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8. If a, b, c be positive real numbers, not all equal, and n is a negative rational number, prove that

$$a^n(a-b)(a-c)+b^n(b-a)(b-c)+c^n(c-a)(c-b) > 0.$$

Solution: Let a, b, c be positive real numbers, not all equal and n be a negative rational number.

Case I: Let $a = b \neq c$. Then
$a^n(a-b)(a-c)+b^n(b-a)(b-c)+c^n(c-a)(c-b) = 0+0+c^n(c-a)^2 > 0$.

Case II: Let $a \neq b \neq c$. Without loss of generality, suppose $a > b > c$.
We have $a^n(a-b)(a-c)+b^n(b-a)(b-c) \implies (a-b)[a^n(a-c)-b^n(b-c)]$.

Now, $a > b \implies a^n < b^n \, [\because n < 0]$ and $a > b \implies a - c > b - c \implies c - a > c - b$.

So

$$\begin{aligned} a^n(c-a) < b^n(c-b) &\implies a^n(a-c) > b^n(b-c) \\ &\implies a^n(a-c) - b^n(b-c) > 0 \\ &\implies (a-b)[a^n(a-c) - b^n(b-c)] > 0 \end{aligned} \tag{14}$$

Also $c^n(c-a)(c-b) > 0$.

Hence $a^n(a-b)(a-c) + b^n(b-a)(b-c) + c^n(c-a)(c-b) = 0 + 0 + c^n(c-a)^2 > 0$.

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9. If n be a positive integer greater than 2, prove that $(n!)^2 > n^n$.

Solution: Let r be a positive integer such that $1 < r < n$. Then

$$\begin{aligned} r-1 > 0, n-r > 0 &\implies (n-r)(r-1) > 0 \\ &\implies nr - r^2 - n + r > 0 \\ &\implies r(n-r+1) > n \end{aligned}$$

For $r = 2, 3, \cdots, n-1$, we get

$$\begin{aligned} &2 \cdot (n-1) > n \\ &3 \cdot (n-2) > n \\ &\cdots \\ &(n-1) \cdot 2 > n \end{aligned}$$

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Multiplying these we get

$$\begin{aligned} &[2\cdot 3\cdots(n-2)(n-1)]^2 > n^{n-2} \\ \implies &[2\cdot 3\cdots(n-2)(n-1)]^2\cdot n^2 > n^{n-2}\cdot n^2 \\ \implies &[2\cdot 3\cdots(n-2)(n-1)n]^2 > n^n \\ \implies &(n!)^2 > n^n. \end{aligned}$$

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10. If $a_1, a_2, \ldots, a_n$ are positive reals in Arithmetic Progression, prove that $a_1a_2\ldots a_n > (a_1a_n)^{n/2}$.

Solution: Without any loss of generality, we may assume that $a_1 < a_2 < \cdots < a_n$. Then $a_n = a_1 + (n-1)d$, where $d =$ common difference.

Now,

$$\begin{aligned} a_1a_n - a_2a_{n-1} &= a_1[a_1 + (n-1)d] - (a_1+d)[a_1+(n-2)d] \\ &= a_2 + a_1(n-1)d - a_1^2 - a_1d - a_1(n-2)d - (n-2)d^2 \\ &= -(n-2)d^2 \\ &< 0. \end{aligned}$$

$$\begin{aligned} \implies &\ a_1a_n - a_2a_{n-1} < 0 \\ \implies &\ a_2a_{n-1} - a_1a_n > 0 \\ \implies &\ a_2a_{n-1} > a_1a_n \end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned} &a_3a_{n-2} > a_1a_n, \\ &\ldots \\ &a_{n-2}a_3 > a_1a_n, \\ &a_{n-1}a_2 > a_1a_n \end{aligned}$$

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Multiplying these,

$$(a_2a_3\cdots a_{n-1})^2 > (a_1a_n)^{n-2} \implies a_1a_2a_3\cdots a_{n-1}a_n > (a_1a_n)^{\frac{n}{2}}.$$

Alternatively: Consider

$$(a_1a_2\ldots a_n)^2 > (a_1a_n)^n,$$

which is equivalent to

$$(a_1a_n)^2(a_2a_{n-1})^2\ldots > (a_1a_n)^n.$$

So we must show that for all $i \in \{2, \ldots, \frac{n+1}{2}\}$,

$$a_ia_{n-i+1} > a_1a_n.$$

$$a_ia_{n-i+1} = (a_1+(i-1)d)(a_1+(n-i)d) = a_1^2 + a_1(i-1)d + a_1(n-i)d + (i-1)(n-i)d^2$$

We have $(i-1)(n-i)d^2 = C > 0$, which implies that

$$a_ia_{n-i+1} = a_1^2 + a_1(n-1)d + C > a_1^2 + a_1(n-1)d = a_1(a_1+(n-1)d) = a_1a_n,$$

so since this inequality holds, then $(a_1a_2\ldots a_n)^2 > (a_1a_n)^n$ holds and by taking the square root of both sides we get the desired result.

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11. If a, b, x, y be positive real numbers, prove that $\dfrac{ax+by}{a+b} \geq \dfrac{(a+b)xy}{ay+bx}$.

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Solution:

$$\begin{aligned}\frac{ax+by}{a+b}-\frac{(a+b)xy}{ay+bx} &= \frac{(ax+by)(ay+bx)-[(a+b)xy](a+b)}{(a+b)(ay+bx)}\\ &= \frac{a^2xy+aby^2+abx^2+b^2xy-(a+b)^2xy}{(a+b)(ay+bx)}\\ &= \frac{a^2xy+aby^2+abx^2+b^2xy-(a^2+2ab+b^2)xy}{(a+b)(ay+bx)}\\ &= \frac{aby^2+abx^2-2abxy}{(a+b)(ay+bx)}\\ &= \frac{ab(x-y)^2}{(a+b)(ay+bx)}\\ &\geq 0\,[\because a,b,x,y>0] \text{ equality holds when } x=y\end{aligned}$$

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12. Prove that $1!\cdot 3!\cdot 5!\cdots(2n-1)!>(n!)^n$.

Solution: We have ${}^{2n}C_r=\dfrac{(2n)!}{r!(2n-r)!}$.

The greatest value of ${}^{2n}C_r$ is ${}^{2n}C_n=\dfrac{(2n)!}{n!n!}$.

$\therefore \dfrac{(2n)!}{r!(2n-r)!}<\dfrac{(2n)!}{n!n!} \implies r!(2n-r)!>(n!)^2$.

Putting $r=1,2,\cdots,(2n-1)$,

$$\begin{aligned}&1!(2n-1)!>(n!)^2\\ &3!(2n-3)!>(n!)^2\\ &\cdots\\ &(2n-1)!1!>(n!)^2\end{aligned}$$

Multiplying these results we gets

$[1!3!\cdots(2n-1)!]^2>(n!)^{2n}$ or $1!3!\cdots(2n-1)!>(n!)^n$.

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13. If a,b,c be positive real numbers, not all equal, prove that

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(a) $2(a^3+b^3+c^3) > a^2(b+c)+b^2(c+a)+c^2(a+b) > 6abc$;

(b) $\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2}+\frac{a+b}{a^2+b^2} < \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$;

(c) $(a^3+b^3+c^3)(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}) > 9$;

(d) $\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} < \frac{a+b+c}{2}$.

Solution:

(a) Applying $AM \geq GM$ for the positive integers $a^2b, a^2c, b^2a, b^2c, c^2a, c^2b$ we get

$$\frac{a^2b+a^2c+b^2a+b^2c+c^2a+c^2b}{6} \geq \sqrt[6]{a^6b^6c^6}$$

$$\Longrightarrow a^2b+a^2c+b^2a+b^2c+c^2a+c^2b \geq 6abc. \tag{16}$$

We know that

$$\frac{a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}}{n} > \frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\cdot\frac{a_1^q+a_2^q+\cdots+a_n^q}{n},$$

where p, q are rational numbers with $pq > 0$.

Taking $p=1, q=2$,

$$\begin{aligned}\frac{a^3+b^3}{2} &> \frac{a+b}{2}\cdot\frac{a^2+b^2}{2}\\ &> \frac{a+b}{2}\cdot\sqrt{a^2b^2}\,[\because AM > GM]\\ &= \frac{ab(a+b)}{2}\end{aligned}$$

$$\Longrightarrow a^3+b^3 > ab(a+b)$$

$$\Longrightarrow a^3+b^3 > a^2b+ab^2 \tag{17}$$

Similarly, we get

$$b^3+c^3 > b^2c+bc^2 \tag{18}$$

$$c^3+a^3 > c^2a+ca^2 \tag{19}$$

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Adding (17),(18) & (19),

$$2(a^3+b^3+c^3) > a^2(b+c)+b^2(c+a)+c^2(a+b) \tag{20}$$

From (16) & (eq1.19) we have

$$2(a^3+b^3+c^3) > a^2(b+c)+b^2(c+a)+c^2(a+b) > 6abc$$

(b)

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)-\frac{a+b}{a^2+b^2} &= \frac{a+b}{2ab}-\frac{a+b}{a^2+b^2}\\ &= (a+b)\left[\frac{a^2+b^2-2ab}{2ab(a^2+b^2)}\right]\\ &= \frac{(a+b)(a-b)^2}{2ab(a^2+b^2)}\\ &> 0.\end{aligned}$$

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$$\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) < \frac{a+b}{a^2+b^2} \tag{21}$$

$$\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right) < \frac{b+c}{b^2+c^2} \tag{22}$$

$$\frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}\right) < \frac{c+a}{c^2+a^2} \tag{23}$$

Adding (21), (22) & (23), we get

$$\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2}+\frac{a+b}{a^2+b^2} < \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

(c) We know that if a, b, c are positive real numbers, not all equal and p, q are rational numbers with opposite signs, then

$$\frac{a^p+b^p+c^p}{3}\cdot\frac{a^q+b^q+c^q}{3} > \frac{a^{p+q}+b^{p+q}+c^{p+q}}{3}$$

Put $p = 3, q = -3$, we get

$$\frac{a^3+b^3+c^3}{3}\cdot\frac{a^{-3}+b^{-3}+c^{-3}}{3} > 1$$
$$\implies (a^3+b^3+c^3)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) > 9.$$

(d)

$$\frac{a+b}{4}-\frac{ab}{a+b}=\frac{(a+b)^2-4ab}{4(a+b)}=\frac{(a-b)^2}{4(a+b)} > 0.$$
$$\implies \frac{ab}{a+b} < \frac{a+b}{4} \tag{24}$$

Similarly,

$$\frac{bc}{b+c} < \frac{b+c}{4} \tag{25}$$

$$\frac{ca}{c+a} < \frac{c+a}{4} \tag{26}$$

Adding (24), (25) & (26),

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} < \frac{a+b+c}{2}.$$

$***$

14. If a, b, c, d be all real numbers, prove that $(a^2+b^2+c^2+d^2)(a^2+b^2)(c^2+d^2) \geq (abc+bcd+cda+dab)^2$.

Solution: By Cauchy-Schwarz inequality, we get

$$[a(bc)+b(cd)+c(da)+d(ab)]^2 \leq (a^2+b^2+c^2+d^2)[(bc)^2+(cd)^2+(da)^2+(ab)^2]$$
$$= (a^2+b^2+c^2+d^2)(b^2+d^2)(c^2+d^2).$$

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15. If a, b, c, x, y, z be all real numbers and $a^2 + b^2 + c^2 = 1, x^2 + y^2 + z^2 = 1$, prove that $-1 \leq ax + by + cz \leq 1$.

Solution: By Cauchy-Schwarz inequality, we get

$$\begin{aligned}(ax + by + cz)^2 &\leq (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \\ &= 1.1 \, [\because a^2 + b^2 + c^2 = 1 \& x^2 + y^2 + z^2 = 1]\end{aligned}$$

$\implies (ax + by + cz)^2 \leq 1.$

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16. If $a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d_1, d_2, d_3$ be all real numbers, prove that

$$(a_1b_1c_1d_1 + a_2b_2c_2d_2 + a_3b_3c_3d_3)^4 \leq (a_1^4 + a_2^4 + a_2^4)(b_1^4 + b_2^4 + b_3^4)(c_1^4 + c_2^4 + c_3^4)(d_1^4 + d_2^4 + d_3^4).$$

Solution: By Cauchy-Schwarz inequality, we get

$\left[\{(a_1b_1)(c_1d_1) + (a_2b_2)(c_2d_2) + (a_3b_3)(c_3d_3)\}^2\right]^2$
$\leq [a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2]^2 \cdot [c_1^2d_1^2 + c_2^2d_2^2 + c_3^2d_3^2]^2$
$\leq (a_1^4 + a_2^4 + a_3^4)(b_1^4 + b_2^4 + b_3^4)(c_1^4 + c_2^4 + c_3^4)(d_1^4 + d_2^4 + d_3^4).$

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Exercises 1B

1. If a, b, c be positive real numbers, prove that

   (a) $a^4 + b^4 + c^4 \geq abc(a + b + c)$,

   (b) $(\frac{a + b + c}{3})^3 \geq a(\frac{b + c}{2})^2$,

   (c) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$,

   (d) $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$,

   (e) $\frac{9}{a + b + c} \leq \frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

   (f) $\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} > \frac{3}{2}$, unless $a = b = c$

   (g) $(ab + bc + ca)(ab^{-1} + bc^{-1} + ca^{-1}) \geq (a + b + c)^2$.

Solution:

(a) We have $\frac{a^{p+q} + b^{p+q} + c^{p+q}}{3} \geq \frac{a^p + b^p + c^p}{3} \cdot \frac{a^q + b^q + c^q}{3}$

If p, q are rational numbers the equality occurs when $a = b = c$. Take $p = 3, q = 1$, we get $\frac{a^4 + b^4 + c^4}{3} \geq \frac{a^3 + b^3 + c^3}{3} \cdot \frac{a + b + c}{3}$ the equality occurs when $a = b = c$.

Again we know that $\frac{a^3 + b^3 + c^3}{3} \geq (a^3b^3c^3)^{1/3} = abc \Rightarrow \frac{a^4 + b^4 + c^4}{3} \geq abc(\frac{a + b + c}{3})$

i.e $a^4 + b^4 + c^4 \geq abc(a + b + c)$ the equality occurs when $a = b = c$.

(b)

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